HIGHER - SPIN GRAVITY

An overview

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Mostly based on [X Bekaert, N B. and P. Sundell, Rev. Mod Phys 84 (2012) "How higher-spin gravity surpasses the spin two barrier"].

- At dawn of QFT . Majorana (1932), Dirac (1936), Fierz-Pauli (1939), and most notably Wigner's 1939 classification of UIR's of Poincaré group $ISO(3,1)$.

. Relativistic, linear & covariant equations : Bargmann-Wigner (1948)

$\longrightarrow$ massless, helicity particles characterized by

- Mass $m=0$; • helicity $s \in \{0, 1/2, 1, 3/2 \cdots\}$

[• Rem : In the $m=0$ case, there are also the "continuous" or "infinite" spin UIR's $\rightsquigarrow$ $\vec{\mu} \neq \vec{0}$ in $\mathbb{R}^{D-2}$.]

- Appearance of NO-GO results

↳ Problems with .

- Minimal $u(1)$ coupling for $s \geq 3/2$ (1961)
- Minimal Lorentz coupling for $s \geq 5/2$ (1964)
- Infinite-component Majorana-like equations (1968)

  (tachyons)

↳ Together with the observation of high-spin hadronic resonances

Belief that consistent high-spin interactions require infinitely-many fields of unbounded spin .

Once the HS representations have been seen to exist in the sense of UIR's of spacetime isometry algebra, i.e. first quantization, then standard second quantization naturally requires a covariant Lagrangian.

Fierz-Pauli program

Associate a quadratic, local and covariant Lagrangian to every UIR of maximally-symetric spacetime-isometry algebra

- Initiated by F.P in 1939 for massive, spin-2 particle in $\mathbb{R}^{1,3}$. Then, notably [Chang (67), Schwinger (70), Singh-Hagen (74)]

- In 1978, Fronsdal and Fang gave Lagrangian for $m = 0$ helicity-s field around $\mathbb{R}^{1,3}$ and $(A)dS_4$ by taking the $m \to 0$ limit of Singh-Hagen's $\mathcal{L}$ and introducing (bosons) $\varphi_{\mu_1 \ldots \mu_s}(x)$ and gauge parameter $\epsilon_{\mu_1 \ldots \mu_{s-1}}(x)$ subject to trace constraints:

$$\bar{g}^{\mu\nu} \bar{g}^{\rho\sigma} \varphi_{\mu\nu\rho\sigma\ldots} \equiv 0 \equiv \bar{g}^{\mu\nu} \epsilon_{\mu\nu\rho\ldots}$$

$\boxed{\text{Spin-}s\text{, massive fields}}$: $\quad \varphi_{\mu_1 \ldots \mu_s} = \varphi_{(\mu_1 \ldots \mu_s)}$

and massless

$$\begin{cases} (\Box - m^2)\, \varphi_{\mu_1 \ldots \mu_s} = 0 \\ \partial^\mu \varphi_{\mu \nu_2 \ldots \nu_s} = 0 \\ \eta^{\mu\nu} \varphi_{\mu\nu\alpha_3 \ldots \alpha_s} = 0 \end{cases}$$

If $m = 0$, action of $\boxed{\text{gauge symmetries}}$

$$\delta_\epsilon \varphi_{\mu_1 \ldots \mu_s} = \partial_{(\mu_1} \epsilon_{\mu_2 \ldots \mu_s)}$$

to remove extra degrees of freedom.

- Examples : 1) spin-1

$$\begin{cases} \Box A_\mu = 0 \\ \partial^\mu A_\mu = 0 \\ \delta_\epsilon A_\mu = \partial_\mu \epsilon(x) \end{cases}$$

Maxwell's field

↳ Equivalent description : $\begin{cases} \partial^\mu F_{\mu\nu} = 0, \\ \partial_{[\mu} F_{\nu\rho]} = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} F_{\mu\nu} = 2\, \partial_{[\mu} A_{\nu]} \\ \Box F_{\mu\nu} = 0 \end{array}}$ Faraday

$\boxed{\text{Example}}$ 2) Spin-2

$$\begin{cases} (\Box - m^2)\, h_{\mu\nu} = 0 \\ \partial^\mu h_{\mu\nu} = 0 \\ \eta^{\mu\nu} h_{\mu\nu} = 0 \end{cases} \xrightarrow[m=0]{} \quad \delta_\epsilon h_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$$

↳ Equivalent description : $\eta^{\alpha\beta} \overset{(1)}{R}_{\alpha\mu\nu\beta} = 0$ i.e. $\overset{(1)}{R}_{\mu\nu}(h) = 0$

linearised Einstein's equation

$$\partial_{[\mu} \overset{(1)}{R}_{\nu\rho]\alpha\beta} = 0 \qquad \partial^\mu \overset{(1)}{R}_{\mu\nu\alpha\beta} = 0$$

$\Rightarrow \quad \Box \overset{(1)}{C}_{\mu\nu\alpha\beta} = 0$ Linearised Weyl tensor IS the massless, spin-2 field

⇝ gravitat. wave

- In the general, spin-s case.

$$\Box\, C_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} = 0$$

where $C_{\mu_1 \dots \mu_s, \nu_1 \dots \nu_s} = \partial_{\mu_1} \cdots \partial_{\mu_s} \varphi_{\nu_1 \dots \nu_s} + \dots$

traceless w.r.t. $\eta_{\mu\nu}$

- Maxwell's theory : $A_\mu(x) := \varphi_\mu(x)$, $\delta_\epsilon A_\mu(x) = \partial_\mu \epsilon(x)$

  - $S[A_\mu] = -\frac{1}{4}\int d^4x \; F^{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu} := 2\,\partial_{[\mu} A_{\nu]}$

  - $\delta_\epsilon S[A_\mu] = 0 \iff \partial^\mu F_{\mu\nu} \equiv 0$ (Noether id.)

- Fierz-Pauli in metric-like notation :

  $h_{\mu_1\mu_2}(x) =: \varphi_{\mu(2)}(x)$, $\delta_\epsilon \varphi_{\mu(2)} = 2\,\partial_\mu \epsilon_\mu$ $\left(\delta_\epsilon h_{\mu\nu} = 2\,\partial_{(\mu}\epsilon_{\nu)}\right)$

  - $S_0[\varphi_{\mu(2)}] = -\frac{1}{2}\int d^4x \left[\partial^\nu \varphi^{\mu(2)}\,\partial_\nu \varphi_{\mu(2)} + \dots\right]$

  - $\delta_\epsilon S_0[\varphi_{\mu(2)}] = 0 \iff \partial^\mu \overset{(1)}{G}_{\mu\nu}(x) \equiv 0$, $\overset{(1)}{G}_{\mu\nu} := \overset{(1)}{R}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\overset{(1)}{R}$.

- Fronsdal's formulation

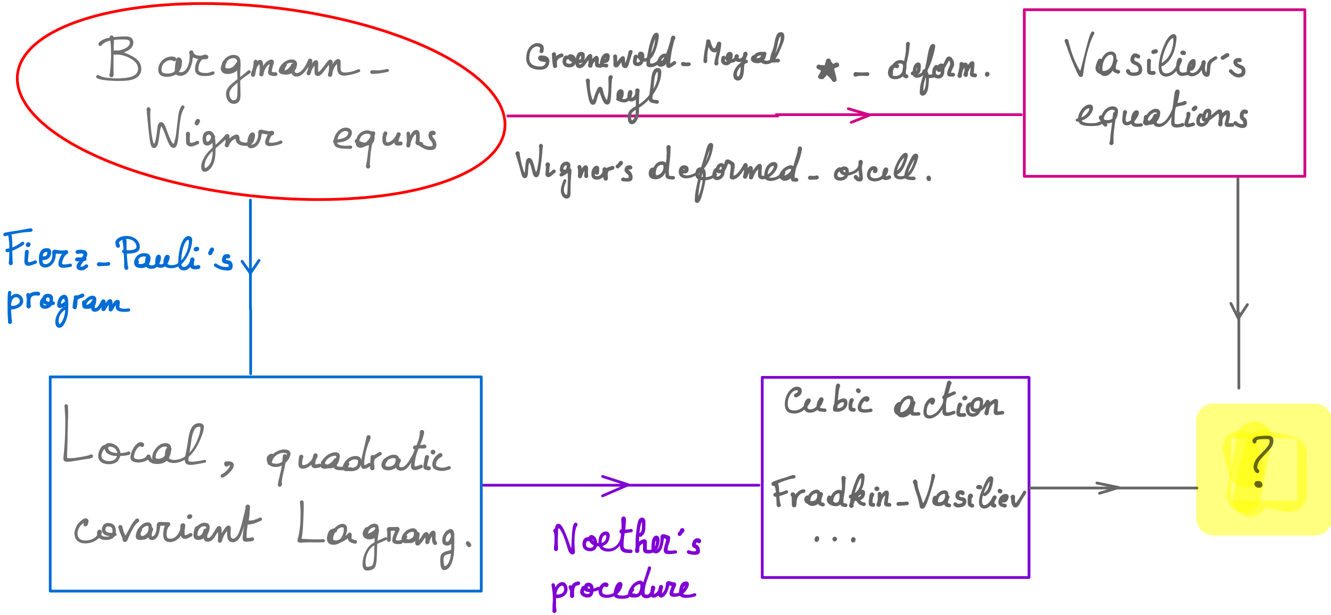
- $$\varphi_{\mu_1\dots\mu_s} = \varphi_{(\mu_1\dots\mu_s)} = \varphi_{\mu(s)} \,,$$

  ↳ Gauge transformation : $\delta_\epsilon \varphi_{\mu_1\dots\mu_s} = s\,\bar\nabla_{(\mu_1}\epsilon_{\mu_2\dots\mu_s)}$

Constr. : $\bar g^{\mu\nu}\bar g^{\rho\sigma}\varphi_{\mu\nu\rho\sigma\dots} \equiv 0 \quad (s \geqslant 4)\,, \qquad \bar g^{\mu\nu}\epsilon_{\mu\nu\dots} \equiv 0 \quad (s \geqslant 3)$

- $$S^{Fr}[\varphi] = \int \mathcal{L}(\varphi, \bar\nabla\varphi)\,, \qquad \frac{\delta S^{Fr}}{\delta\varphi_{\mu(s)}} =: G^{\mu(s)} \approx 0$$

  $$\nabla^{\mu_1} G_{\mu_1\mu_2\dots\mu_s} \propto \bar g_{(\mu_2\mu_3}\nabla^\alpha G'_{\mu_4 \quad \mu_s)\alpha}$$ Noether identity



$\nabla \rightarrow \nabla + A$ quasi-minimal coupling

around AdS ✓

- **WAVE EQUATIONS $\rightsquigarrow$ BARGMANN-WIGNER IN AdS** [Fronsdal in 70's].

Conventions and notation for $so(2,d)$

↳ Lie algebra $so(2,d)$ with generators $M_{AB} = M_{AB}^{\dagger}$

$$[M_{AB}, M_{CD}] = i\left(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}\right)$$

$A, B, . = 0', 0, 1, \ldots, d$ $\qquad\qquad$ $a, b, .. = 0, 1, \ldots, d$

$$\eta_{AB} = \mathrm{diag}(\underset{0'}{-}, \underset{0}{-}, \underset{1}{+}, \underset{\cdots}{\ldots}, \underset{d}{+}) \qquad \eta_{ab} = \mathrm{diag}(\underset{0}{-}, \underset{1}{+}, \underset{\cdot}{\ldots}, \underset{d}{+})$$

$$\boxed{P_a = M_{0'a}} \,, \qquad \text{in particular} \quad E = P_0 = M_{0'0}$$

$$\Rightarrow \ [P_a, P_b] = [M_{0'a}, M_{0'b}] = i\,\sigma\, M_{ab} \,, \qquad \sigma = \begin{cases} +1 & AdS_{d+1} \\ -1 & dS_{d+1} \end{cases}$$

$$\boxed{L^{\pm}_i := M_{i0} \mp i\, M_{i0'}} \qquad i = 1, \dots, d \qquad so(d) \text{ index}$$

- $so(2,d) \supset \underbrace{so(2)}_{E} \oplus \underbrace{so(d)}_{M_{ij}}$ maximal compact

$$\boxed{\begin{aligned} &[M_{ij}, M_{kl}] = 4i\,(\delta_{jk} M_{il} + 3 \text{ terms}) \\ &[E, L^{\pm}_i] = \pm L^{\pm}_i \,, \qquad [M_{ij}, L^{\pm\, k}] = 2i\, \delta^{k}_{[j} L^{\pm}_{i]} \\ &[L^{-}_i, L^{+}_j] = 2\,\delta_{ij}\, E + 2i\, M_{ij} \end{aligned}}$$

• $C_2[so(2,d)] := \frac{1}{2} M^{AB} M_{AB}$, $\quad P^2 := P_a \; P_b \; \eta^{ab} = M_{0'a} \, M_{0'b} \, \eta^{ab}$

$$\Leftrightarrow \; P^2 = -\frac{1}{2} M^{AB} M_{AB} + \frac{1}{2} M^{ab} M_{ab} = -C_2[so(2,d)] + C_2[so(1,d)] \, .$$

• On the other hand, using the decomposition $M_{AB} \sim \{ M_{ij} \, , E \, , L^{\pm}_i \}$,

one finds $C_2[so(2,d)] = E\,(E-d) - L^+_i L^{-i} + C_2[so(d)]$, so that

$$\boxed{C_2\left[so(2,d) | \mathfrak{D}(e_0,\vec{s})\right] = -e_0(-e_0+d) + s_1(s_1+d-2) + \dots + s_r(s_r+d-2r)}$$

upon evaluation on Lowest_weigth state $|e_0, \vec{s}\rangle_{i\dots j\dots}$ obeying

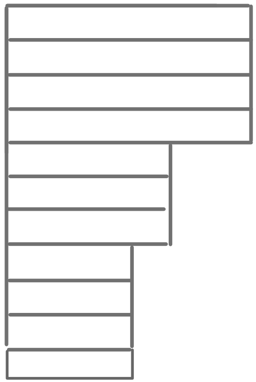
$$(E-e_0)|e_0,\vec{s}\rangle_{i\dots j\dots} = 0 = L^-_k \, |e_0,\vec{s}\rangle_{i\dots j\dots}$$

Where the vacuum $|e_0, \vec{s}\rangle_{i\ldots j\ldots}$ of the generalized Verma module

$$g(e_0, \vec{s}) := \left\{ L^+_{i_1} \ldots L^+_{i_n} \, |e_0, \vec{s}\rangle_{i\ldots j\ldots} \right\}_{n=0,1,2,\ldots}$$

with highest $so(2,d)$-weight $\Lambda = (-e_0, \vec{s})$

transforms in the $so(d)$-irrep $\mathcal{R}^{so(d)}_{\vec{s}}$ associated with Young diagram

$\vec{s} =$  of $so(d)$.

. Taking the wave-equation representation $\rho(P_a) = -i\,\ell\,\nabla_a$

with $\ell^2 = \lambda^{-2} \propto$ square radius of AdS_{d+1},

$-m^2c^2 = -E^2/c^2 + \vec{p}^{\,2}c^2$

$E^2 = m^2c^4 + \vec{p}^{\,2}c^2$

$p^2 = -\hbar^2\Box = -c^2m^2$

$$P_a = \left(-\frac{E}{c}, \vec{p}\right) = -i\hbar\,\overbrace{\left(\tfrac{1}{c}\partial_t, \vec{\nabla}\right)}^{\nabla_a}, \quad \Box = \eta^{ab}\nabla_a\nabla_b \rightsquigarrow \left(\Box - \frac{c^2}{\hbar^2}m^2\right)\phi = 0$$

$P^2 := P^a P_a = -\hbar^2\Box$

Measure masses in unit of λ : $m \to \frac{\hbar}{c}\lambda\,\bar{m}$ where $\bar{m}$ dimensionless

$L = M^{-1}\cdot(\hbar/c)$

$$\rho(P^2) = -\ell^2\,\nabla_a\nabla_b\,\eta^{ab}, \quad \text{hence} \quad (\Box - \lambda^2\,\bar{m}_h^2)\,\phi_h = 0$$

entails $-P^2 = \ell^2\,\bar{m}_h^2$, so that

$$\boxed{\ell^2\,\bar{m}_h^2 = C_2\left[so(2,d)\,|\,\mathcal{D}(e_0,\vec{s})\right] - C_2\left[so(1,d)\,|\,\underset{\sim}{h}\right]}$$

where $\underset{\sim}{h}$ denotes the $so(1,d)$ Lorentz type of the tensor ϕ_h in the Lorentz-covariant realisation of the abstract UIR $\mathcal{D}(e_0, \vec{s})$.

$$\boxed{\begin{array}{lcl} \text{Wigner} & \longrightarrow & \text{Bargmann - Wigner} \\ \mathcal{D}(e_0, \vec{s}) & \longrightarrow & \{ \phi_{h_\alpha} \quad \} \; (\Box - \lambda^2 \bar{m}^2_{h_\alpha}) \, \phi_{h_\alpha} = 0 \} . \end{array}}$$

Example : Spin-s UIR of $so(2,3) \rightsquigarrow AdS_4$ [Fronsdal , 70's]

$\vec{s}$ of $so(3)$: $\boxed{\quad s \quad}$ $\rightsquigarrow$ $|e_0, \vec{s}\rangle_{i_1 \dots i_s}$ vacuum.

$C_2[so(3) \mid \vec{s}\,] = s(s+d-2) = s(s+1)$

$e_0 = s+1$ leading to $L^{+i} |s+1, s\rangle_{i \dots}$ being nul. $(s > 0)$

$\bullet\ C_2\left[so(2,3)\,|\,\mathcal{D}(s+1,s)\right] = -e_0(-e_0+3) + s(s+1) = 2(s+1)(s-1)$

↳ The set of Lorentz tensors carrying this UIR $\mathcal{D}(s+1,s)$ is

$$\{\phi_{h_\alpha}\} = \left\{ C^{a(s+k),\,b(s)} \sim \begin{array}{|c|c|}\hline s & k \\ \hline s & \\ \cline{1-1}\end{array} \right\}$$

with

$$C^{a(s+k),\,b(s)} := C^{a_1 a_2 \cdots a_{s+k},\, b_1 b_2 \cdots b_s} \equiv C^{(a_1 a_2 \cdots a_{s+k}),(b_1 b_2 \cdots b_s)}$$

s.t.

$$\bullet\ C^{(a_1 a_2 \cdots a_{s+k},\, b_1) b_2 \cdots b_s} \equiv 0 .$$

$$\bullet\ C^{a(s+k),\,b(s)} \quad \eta^{ab}\text{-traceless}$$

• The Lorentz tensors $\{ C^{a(s+k),\, b(s)} \}_{k=0,1,\dots}$ obey the

linear, relativistic wave equations ($\rightsquigarrow$ BW program)

$$\boxed{\left(\Box - \lambda^2 M^2_{(s,k)} \right) C^{a(s+k),\, b(s)}(x) = 0}$$

where

$$M^2_{(s,k)} = -\sigma \left[4\epsilon_0 + 2s + k(k+2s+2\epsilon_0+1) \right]$$

$(\sigma = +1 : AdS)$

$$\epsilon_0 := \frac{(d-2)}{2}$$

• In particular, for $d=3$, $k=0$, the primary Weyl tensors obey

$$\boxed{\left(\Box + 2\lambda^2 (s+1) \right) C_{a(s),\, b(s)} = 0 \,.}$$

check. $M^2 = C_2[SO(2,3)] - C_2[SO(1,3) \mid (s,s)] = 2(s+1)(s-1) - \left(s(s+4-2) + s(s+4-4) \right)$

$$= -2(s+1) \,.$$

Note. In the scalar case $s = 0$, the primary Weyl tensor

$\phi(x)$ obeys $(\Box + 2\lambda^2)\,\phi(x) = 0$ in AdS_4, where

$$\bar{M}^2_{(0,0)} = -2 = C_2[so(2,3)\,|\,\mathcal{D}(e_0, 0)] = -e_0(-e_0+3)$$

leaving 2 possibilities compatible with unitarity:

$\hookrightarrow$ $e_0 = 1$ (Dirichlet) or 2 (Neuman) BC's.

. So, in the zoology of "massless" UIR's

$\rightsquigarrow$ (bosonic) fields propagating in AdS_4, we have

$\mathcal{D}(s+1, s)$ $s = 0, 1, 2, \ldots$ and $\mathcal{D}(2, 0)$. Fronsdal on-shell fields

Dirac singletons and Flato-Fronsdal

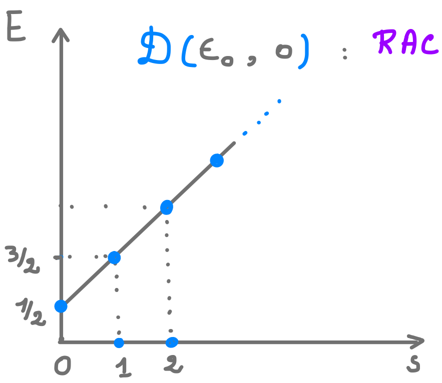
$\left[\epsilon_0 = \frac{d-2}{2} \right]$.

Two remarkable $so(2,d)$-UIRs : $\mathcal{D}(\epsilon_0, 0)$ & $\mathcal{D}(\epsilon_0 + \frac{1}{2}, \frac{1}{2})$

Not propagating inside AdS_{d+1} but at ∂AdS_{d+1} .

↳ single line in compact weight space

$\mathcal{D}(\epsilon_0, 0)$: RAC & $\mathcal{D}(\epsilon_0 + \frac{1}{2}, \frac{1}{2})$: DI



Flato – Fronsdal theorem ($d=3$)

- $\mathcal{D}(\tfrac{1}{2},0) \otimes \mathcal{D}(\tfrac{1}{2},0) \simeq \bigoplus_{s=0}^{\infty} \mathcal{D}(s+1,s)$

- $\mathcal{D}(1,\tfrac{1}{2}) \otimes \mathcal{D}(1,\tfrac{1}{2}) \simeq \mathcal{D}(2,0) \oplus \bigoplus_{s=1}^{\infty} \mathcal{D}(s+1,s)$

Consequence : Compositeness of massless particles in AdS_4

RAC : $\Box_3 \, \phi(x) = 0 \quad (*)$ with $\dim(\phi) = \frac{1}{2}$ $\quad \left(\int d^3x \, \partial\phi \cdot \partial\phi \right)$

conformal scalar

- Symmetries of $(*)$: $\dfrac{\mathcal{U}(so(2,d))}{\mathrm{Annih}(\mathrm{RAC})} \simeq \mathcal{A}$ associative algebra

$$\mathcal{A} \xrightarrow{[\cdot\,,\,\cdot]} hs(d+1)$$

$\boxed{\text{Unfolded version and extension of BW}}$ → First-order differential equations

• $\Omega := (h^a, \varpi^{ab})$ $so(2,d)$-valued 1-form $\left(\Omega = dx^\mu \Omega_\mu{}^{AB} \tfrac{1}{2} M_{AB}\right)$

• AdS_{d+1} : $R_0 := d\Omega + \Omega\,\Omega = 0$ & invertibility of $h^a{}_\mu$

• $\nabla = d + \varpi$ Lorentz-cov. derivative $\left(\nabla^2 V^a = -\lambda^2 h^a{}_\wedge h_b V^b\right)$

$$(s)\left\{\begin{array}{l}
\nabla \overset{(0)}{C}{}_{a(s),b(s)} = h^c \overset{(1)}{C}{}_{c\{a(s),b(s)\}} \\
\nabla \overset{(1)}{C}{}_{a(s+1),b(s)} = h^c \overset{(2)}{C}{}_{c\{a(s+1),b(s)\}} + \lambda_1\, h_{\{a} \overset{(0)}{C}{}_{a(s),b(s)\}} \\
\nabla \overset{(2)}{C}{}_{a(s+2),b(s)} = h^c \overset{(3)}{C}{}_{c\{a(s+2),b(s)\}} + \lambda_2\, h_{\{a} \overset{(1)}{C}{}_{a(s+1),b(s)\}} \\
\qquad\vdots \qquad\qquad\qquad \vdots
\end{array}\right.$$

λ_i's $\propto \lambda^2$.

• Due to the symmetry properties of the zero-forms $\overset{(*)}{C}$'s,

the system (S)

(i) reproduces the wave equations

$$\left(\square - \lambda^2 M^2_{(s,k)}\right) C^{a(s+k),\,b(s)}(x) = 0 \;;$$

(ii) Can be integrated so as to give a <u>1-form module</u>

$$\boxed{\begin{aligned}
&\nabla e^{a(s-1)} + h_b\, \omega^{a(s-1),b} = 0\\
&\nabla \omega^{a(s-1),b} + h_b\, X^{a(s-1),bb} + \bar{\lambda}_1\, h^{\{b}\, e^{a(s-1)\}} = 0\\
&\qquad\vdots\\
&\nabla X^{a(s-1),\,b(s-1)} + \bar{\lambda}_{s-1}\, h^{\{b}\, X^{a(s-1),\,b(s-2)\}} = h_a \wedge h_b\, C^{a(s-1),\,b(s-1)}
\end{aligned}}$$

· Gauge symmetries : differential and algebraic

↳ Minimal set of fields & gauge parameters ⇝ Fronsdal.

$$-2\mathcal{L}(\varphi,\nabla\varphi) = \nabla_\nu \varphi_{\mu(s)} \nabla^\nu \varphi^{\mu(s)} - \frac{s(s-1)}{2} \nabla_\nu \varphi'_{\mu(s-2)} \nabla^\nu \varphi'^{\mu(s-2)}$$

$$+ s(s-1) \nabla_\nu \varphi'_{\mu(s-2)} \nabla_\rho \varphi^{\rho\nu\mu(s-2)} - s \nabla_\nu \varphi^\nu{}_{\mu(s-1)} \nabla_\rho \varphi^{\rho\mu(s-1)}$$

$$- \frac{s(s-1)(s-2)}{2} \nabla_\nu \varphi'^\nu{}_{\mu(s-3)} \nabla_\lambda \varphi'^{\lambda\mu(s-3)}$$

$$+ m_e^2 \varphi^{\mu(s)} \varphi_{\mu(s)} + m_e'^2 \varphi'^{\mu(s-2)} \varphi'_{\mu(s-2)} .$$

$$m_e = \lambda^2 \left(s^2 + (D-6)s - 2D + 6 \right)$$

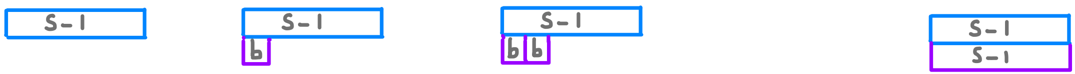
· $\bar{R}_{\mu\nu\rho\sigma} = -\lambda^2 (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\nu\rho} \bar{g}_{\mu\sigma})$, $\qquad \lambda^2 = \dfrac{-2\Lambda}{(D-1)(D-2)}$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

Back to frame-like, Lopatin-Vasiliev's formulation:

↳ Familly of connection 1-forms for spin-s.

$$\left\{ e^{a(s-1)} \, , \; \omega^{a(s-1),b} \, , \; X^{a(s-1),b(2)} \, , \; \dots \, , \; X^{a(s-1),b(s-1)} \right\} .$$



All are Lorentz-valued $so(1,d)$-tensors.

↳ Packed up into a single $so(2,d)$-valued 1-form

$$W^{\begin{array}{|c|}\hline \quad\quad \\ \hline s-1 \\ \hline \end{array}} = dx^\mu \, W_\mu{}^{A(s-1),B(s-1)}$$